Topics in complex analysis   Lecture 12   (Dec 9, 2024)

Lemma 8.11. *Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}^m$ be holomorphic. If $\|f\|_2$ is constant, then f is constant. Here $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{C}^m$.*

Recall the Euclidean norm on $\mathbb{C}^m$ is given by

$$\|x\|_2^2 := \sum_{j=1}^{m} x_j \bar{x}_j .$$

Proof The idea is to differentiate $\|f\|_2^2 = c$ twice.

Consider the differential operator $2\frac{\partial}{\partial \bar z_j} := \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}$

Then

$$0 = \frac{\partial}{\partial \bar z_j} c = \frac{\partial}{\partial \bar z_j} \|f\|_2^2 = \sum_{k=1}^{m} \frac{\partial}{\partial \bar z_j} \left(f_k \bar f_k \right)$$

Leibniz $\to$

$$= \sum_{k=1}^{m} \Bigg(\underbrace{\frac{\partial f_k}{\partial \bar z_j}}_{=0 \text{ (exercise)}} \bar f_k + f_k \frac{\partial \bar f_k}{\partial \bar z_j} \Bigg) = \sum_{k=1}^{m} f_k \frac{\partial \bar f_k}{\partial \bar z_j} \quad (*)$$

$= 0$ (exercise) for every holo fct.

Now consider the differential op $2\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}$

One can check the following identities for every holo fct f:

$$\frac{\partial \bar f}{\partial \bar z_j} = \left(\overline{\frac{\partial f}{\partial z_j}} \right) \quad \text{and} \quad \frac{\partial \bar f}{\partial z_j} = \left(\overline{\frac{\partial f}{\partial \bar z_j}} \right)$$

"Complex conjugation distributes throughout fractions"

Differentiating $(*)$ once again

$$0 = \sum_{k=1}^{m} \frac{\partial}{\partial z_j} \left(f_k \frac{\partial \bar f_k}{\partial \bar z_j} \right) \qquad = \frac{\partial}{\partial z_j} \left(\overline{\frac{\partial f}{\partial z_j}} \right) = \overline{\frac{\partial}{\partial \bar z_j} \left(\frac{\partial f}{\partial z_j} \right)} \quad \text{Holo!}$$

Leibniz $\to$

$$= \sum_{k=1}^{m} \left| \frac{\partial f_k}{z_j} \right|^2 + \sum_{k=1}^{m} f_k \underbrace{\frac{\partial^2 \bar f_k}{\partial z_j \partial \bar z_j}}_{=0}$$

$$= \sum_{k=1}^{n} \left| \frac{\partial f_k}{\partial z_j} \right|^2 = \left\| \frac{\partial f}{\partial z_j} \right\|_2^2$$

Since j was arbitrary and D is connected, this means all f_k are constant. $\square$

Theorem 8.12 (Failure of the Riemann mapping theorem in higher dimensions)**.** *Let $n \geq 2$. Then there exists no biholomorphic map $f : D_1^n(0) \to B_1(0)$, where the ball $B_1(0)$ is defined with respect to the Euclidean metric.*

Proof Assume to the contrary such an f exists.

For a fixed $\omega \in D_1^1(0)$ = open unit Ball in $\mathbb{C}$ we consider the map

$$F_\omega : D_1^{n-1}(0) \longrightarrow \mathbb{C}^n, \quad z' \longmapsto \frac{\partial f}{\partial z_n}(z', \omega)$$

1 in the subscript $\hat{=}$ $\underbrace{(1, \dots, 1)}_{n-1 \text{ times}}$

Claim F_ω can be extended to $\partial D_1^{n-1}(0)$ by 0.

Fix any sequence $(z_j')_{j \in \mathbb{N}}$ in $D_1^{n-1}(0)$ such that $\lim_{j \to \infty} z_j' \in \partial D_1^{n-1}(0)$. Consider the sequence $(f_j)_{j \in \mathbb{N}}$ of functions $f_j : D_1^1(0) \longrightarrow B_1(0)$ with

$$f_j(\omega) := f(z_j', \omega)$$

holo fixed!

Since (f_j) is unif bdd (as $f_j(D_1^1(0)) \subseteq B_1(0)$), Applying Montel's thm to (f_j) yields the existence of a nonrelabeled subsequ of f_j st $f_j \longrightarrow g$ loc. unif on $D_1^1(0)$, for some holomorphic fct $g : D_1^1(0) \longrightarrow \overline{B}_1(0)$.

By construction $\forall \omega \in D_1'(0)$, the sequence $(z_j', \omega)_{j \in \mathbb{N}}$ converges to a point $z_\omega \in \partial_1'' D(0)$. We claim $g(\omega) \in \partial B_1(0)$. If not, we would have

continuous by biholo

$$\partial D_1^n(0) \ni \lim_{j\to\infty} (z_j', \omega) = \lim_{j\to\infty} f^{-1} \circ \underbrace{f(z_j', \omega)}_{\longrightarrow g(\omega)}$$

$$= f^{-1} \circ g(\omega) \in D_1^n(0)$$

$\lightning$

Hence $g(D_1'(0)) \subseteq \partial B_1(0)$. Hence $\|g\|_2 = 1$ on $D_1'(0)$ and thus g is constant by Lem 8.11.

Since g is also holo (as loc unif limit of holo fcts) this yields

$$0 = g'(\omega) = \lim_{j\to\infty} f_j'(\omega) = \lim_j F_\omega(z_j')$$

Thm 1.5

The arbitrariness of z_j' yields the claim.

By the max principle we obtain $F_\omega = 0$.

By def of F_ω this means $\det(Df(z', \omega)) = 0$.

On the other hand since $\boxed{f^{-1} \circ f = \mathrm{Id}}$,

$$\text{Identity matrix} = Df^{-1}(f(z', \omega))\, Df(z', \omega)$$

which forces $\ker Df(z', \omega) = \{0\}$. $\lightning$ $\square$

Lemma 8.15. *Let X be a metric space and I be a set of indices. If for all $i \in I$ the function $u_i : X \to \mathbb{R} \cup \{\pm\infty\}$ is lower semicontinuous, then the function $\bar{u}(x) := \sup_{i \in I} u_i(x)$ is lower semicontinuous. Moreover, a function $u : X \to \mathbb{R} \cup \{\pm\infty\}$ is lower semicontinuous if and only if the set $\{x \in X : u(x) \leq t\}$ is closed for all $t \in \mathbb{R}$*[1].

Proof To show lower semicontinuity of $\bar{u}$, let $(x_n)_{n \in \mathbb{N}}$ be a sequence converging to x. Then

$$\bar{u}(x) = \sup_i u_i(x) \underset{u_i \text{ lsc } \forall i \in I}{\leq} \sup_i \liminf_{n \to \infty} \underbrace{u_i(x_n)}_{\leq \sup_i u_i(x_n) = \bar{u}(x_n)}$$

$$\leq \sup_i \liminf_{n \to \infty} \bar{u}(x_n) = \liminf_{n \to \infty} \bar{u}(x_n).$$

Now we assume u is lsc. Let $t \in \mathbb{R}$ and $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\{u \leq t\}$ converging to $x \in X$.

But

$$u(x) \leq \liminf_{n \to \infty} u(x_n) \leq t \implies \{u \leq t\} \text{ closed.}$$

Vice versa, assume $\{u \leq t\}$ closed $\forall t \in \mathbb{R}$. Fix $x_0 \in X$ and $(x_n)_{n \in \mathbb{N}}$ converging to x_0. If $\liminf_{n \to \infty} u(x_n) = +\infty$ there's nothing to show. Otherwise, up to choosing a subsequence we may and will assume $\liminf_{n \to \infty} u(x_n)$ is actually a limit $u_0 \in \mathbb{R} \cup \{-\infty\}$.

Fix $t > u_0$. Then for n suff large, $x_n \in \{u \leq t\}$. Since the latter is closed, this means $x_0 \in \{u \leq t\}$. So $u(x_0) \leq t$. Letting $t \to u_0$ yields

$$u(x_0) = u_0 = \liminf_{n \to \infty} u(x_n). \qquad \square$$

$+\infty$ excluded!

Definition 8.16. Let $U \subset \mathbb{C}$ be open and $u : U \to \mathbb{R} \cup \{-\infty\}$. We say that u is subharmonic in U if u is upper semicontinuous and for all $z_0 \in U$ there exists $\delta > 0$ such that for all $0 < r < \delta^2$ it holds that

$$u(z_0) \leq \frac{1}{2\pi r} \int_{\partial B_r(z_0)} u(z)\,\mathrm{d}z. \tag{8}$$

Why does the integral make sense?

(1) u is Borel measurable (Lemma 8.15)

(2) u is locally bounded from above by the foll. lemma.

<u>Lemma</u> X metric space, $u : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ usc. Let $K \subseteq X$ be cpt. If u doesn't attain $+\infty$ on K, then u is bdd above on K.

<u>Proof</u> Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in K s.t. $u(x_n) \longrightarrow \sup u(K)$. Wlog we may and will assume $x_n \longrightarrow x \in K$ as $n \longrightarrow \infty$ (K is cpt). Then

$$\sup u(K) = \limsup_{n\to\infty} u(x_n) \leq u(x) < +\infty \qquad \square$$

since u doesn't attain $+\infty$ on K.